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ANDREW RUSSELL FORSYTH
1858-1942

ON 2nd June, Andrew Russell Forsyth passed away. Of his contributions to higher mathematics and the esteem in which he was held by mathematicians all over the world, other journals will bear record. In the *Mathematical Gazette* it is fitting that tribute should be paid to the great work that he did for mathematical teaching in schools.

There can be few teachers in active work today who were teaching at the end of the last century, when most mathematical teaching was academic in character and mainly banished from reality. How few boys then had ever seen a protractor or used a graduated ruler in geometrical work ; algebra for the majority was mere manipulation with symbols that meant nothing to them ; trigonometry was only for the very few, and that again was mostly symbol-grinding. To a gifted few it made an appeal, but few present-day teachers can realise how dull and dreary it was to the majority of pupils.

In the late 1890's Professor John Perry was agitating for root-and-branch reform, but his proposals were so drastic that they were mainly ignored. In 1900 the British Association appointed a committee to consider the reform of school mathematical teaching ; Forsyth was appointed chairman and Perry Secretary. Forsyth can have had but little sympathy with Perry's proposals as actually put forward, but he saw that there was much good in them and, like a wise statesman, he used the energy put out by Perry and directed it so that the proposals were modified and put in such a form that opposition to them became much less violent, and there was much discussion about mathematical reform.

In January 1902 the Mathematical Association appointed its first Teaching Committee. In December 1902, on the motion of Forsyth, the Council of the Cambridge Senate appointed a Syndicate to consider mathematics in the pass examinations of the University. In January 1903 Forsyth became president of the Mathematical Association and for the two years in which he held that office he generally presided at the meetings of the Teaching Committee. From that time on, as a member of the Syndicate and as secretary of the Teaching Committee, I saw much of him ; he was always willing to give his time and help, but the thing that amazed me most was that a man, whose real interest and whose teaching were concerned with the most advanced

mathematical work, should display such a clear and intimate understanding of the difficulties under which the schools were labouring. On the Syndicate, he and the late Professor Hobson did most of the work of persuading the other members of the desirability of reform and, in particular, of freeing the teaching of geometry from the tyranny of Euclid's order. The Syndicate's report was skilfully steered through the Senate by Forsyth, and, as a result, Euclid by degrees ceased to dominate the school teaching of geometry. That in itself was a great achievement, but there were other changes: graphs and the use of 4-figure logarithms were introduced into the Previous Examination and so into most school examinations. It is interesting today to read the account of the discussion of the Syndicate's report in the Senate: several speakers regarded graphs and logarithms as dangerous introductions.

Thus began the great reform of the mathematical teaching in schools throughout the country. Had the Cambridge Senate rejected the Syndicate's report which proposed these changes in the Previous Examination, it is probable that the reformation of mathematical teaching would have been delayed for years.

Some years later Forsyth was a prominent figure in the discussions about abolishing Compulsory Greek at Cambridge and again in the reform of the Mathematical Tripos. He was not a reformer who wished to destroy for the sake of destruction, but he realised that the retention of the order (and of the senior wrangler) in Part I of the Mathematical Tripos was encouraging the best mathematicians at Cambridge to grind too long at the work of Part I instead of pushing on at the proper time to more advanced work: he saw that the adherence to tradition was hampering the work of the day, so he advocated that tradition should go.

Forsyth was a very regular attendant at the annual meetings of the Mathematical Association, and in 1936 he was again President. He had an amazingly clear mind and a wonderful memory. In discussions, and particularly as chairman, he displayed a quiet twinkling humour that prevented angry feelings from arising, and his clear judgment of the weight that should be given to the various arguments for and against any proposal swayed many people in the way they should vote on any question. His kindness and guidance to younger men encouraged many to carry on and improve good work which they had begun.

A great mathematician has passed away: one who has had a wonderful influence on mathematical teaching throughout its whole range. A. W. S.

MEMBERSHIP OF THE ASSOCIATION.

THE total number of members on the roll of the Association at the end of 1941 was 1,752. Unsettled conditions make it impossible to ascertain the number of Associates. A reference to the *Gazette* of February 1940 will show that the fall in membership since the beginning of the war is surprisingly small, and the thanks of the Association are due to its members for their loyal and continued support.

The following members died during 1941 (the list includes some who lost their lives as a result of enemy action): R. Butcher (1939), J. Brill (1884), W. J. Dobbs (1898), I. O. Griffith (1928), E. L. Ince (1926), Miss E. L. Jameson (1910), H. Lob (1922), F. Mayor (1925), J. W. Michell (1925), R. F. Muirhead (1895), T. J. Pearcey (1934), C. Pendlebury (1885), Miss K. M. Rigby (1910), W. Stott (1926). Mr. Butcher lost his life while serving in the Royal Navy.

G. L. P.

THE GEOMETRICAL DERIVATION OF FORMULAS FOR LATERALLY LOADED STRUTS.

By A. J. S. Pippard.

A problem of frequent occurrence in the analysis of engineering structures is the determination of the resultant actions in a uniform bar which is subjected simultaneously to an axial thrust and to lateral loads and terminal couples. This type of problem can be solved by straightforward analytical methods, and well-known formulas for certain cases of loading have been obtained by this means. An alternative method of solution is, however, available by the use of a polar diagram originally due to Ratzersdorfer (1) but discovered independently by H. B. Howard (2) and introduced by him to British engineers. This method is of great value in obtaining graphical solutions in numerical cases and has been used very widely in drawing offices for this purpose. In addition to its use for the solution of specific problems, however, it offers an attractive method of deducing general formulas and is likely to appeal to the imagination of many students far more than the usual approach. This application of the diagram does not appear to have received the attention that it deserves and the object of this paper is to show its value to the teacher. It should be emphasised that while the solution of specific problems involving numerical values of the applied forces requires accurate drawing, a freehand sketch of the diagram is all that is necessary when it is used for the present purpose. For readers who may be unacquainted with the terminology used by engineers it may be said that " encastré " is used to describe the fixing condition when no change of slope is possible at the point of reference and is synonomous with " built-in ".

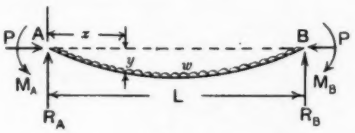

Fig. 1.

In Fig. 1, AB represents a uniform bar of length L which is pin-jointed at the ends and carries an axial thrust P together with end couples M_A and M_B and a uniformly distributed load of intensity w over its whole length.

Following the usual method of solution, we have

$$EI\frac{d^2y}{dx^2} = M = -Py + M', \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where M is the bending moment at a point at a distance x from the origin A, y is the deflexion of this point and M' is the bending moment due to the lateral load and end couples alone. Moments which cause hogging of the bar are taken to be positive and those which cause sagging are taken to be negative.

Differentiating this equation twice, we obtain

$$d^2M/dx^2 + \mu^2 M = w, \text{ where } \mu^2 = P/EI.$$

The form of solution generally used is

$$M = A \sin \mu x + B \cos \mu x + w/\mu^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The constants are evaluated from the terminal conditions. If, instead of a lateral uniformly distributed load the member carries point loads, the term in w disappears but the constants are different in the various sections into which the loads divide the span. Alternatively, the solution can be found in terms of y, and in some cases this is preferable.

Instead of equation (2), however, the alternative form of the solution of the differential equation which is more convenient for polar representation may be used and we write

$$\left.\begin{aligned} M &= C\cos(\mu x-\epsilon)+w/\mu^2 \\ \text{or,}\quad M &= m+w/\mu^2 \end{aligned}\right\}. \qquad \text{.....(3)}$$

The constants to be determined are now C and ϵ.

By differentiating (3), we find

$$dM/dx = S = -\mu C\sin(\mu x-\epsilon), \qquad \text{.....(4)}$$

where S is the shearing force at the point considered.

Also from (1),

$$S = -Pi+S', \qquad \text{.....(5)}$$

where $i = dy/dx$ and is the slope of the bar, while S' is the shearing force due to the lateral loading and terminal couples.

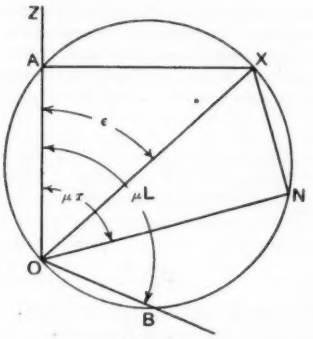

FIG. 2.

Now $m = C\cos(\mu x-\epsilon)$ is the polar equation of a circle and so, translating distances x from the origin in Fig. 1 into angles μx measured from a datum OZ, we set out, in Fig. 2, the line OB at an angle μL to OZ. Then draw $OX = C$ at an angle ϵ to OZ and on OX as diameter describe the circle $OAXB$.

If from O we draw ON at any angle μx to represent a point in the bar at a distance x from the origin, we have

$$ON = OX\cos(\mu x-\epsilon) = C\cos(\mu x-\epsilon),$$

and ON represents the value of m at this point.

Also $$NX = C\sin(\mu x-\epsilon) = -S/\mu.$$

The circle thus gives m and S for all points on the bar.

It may be noted that when $\mu L = \pi$ the value of P is π^2EI/L^2, *i.e.* the Euler critical thrust. Hence in all cases with which we are dealing at present the angle AOB will be less than π.

Positive bending moments are measured outwards from O into the angle AOB and negative moments in the reverse direction. Looking along the positive direction of any vector such as ON the shearing force is positive if its vector NX lies to the right of ON and negative if it lies to the left, as in the figure.

In a specific numerical problem sufficient data will be available to enable the circle to be drawn and values of m can be directly scaled from it. If the bar carries a uniform lateral load this can be shown on the diagram by a circle of radius w/μ^2 with O as centre. The vector difference between the two circles then represents to scale the value of the bending moment at any point, and the diagram is a complete bending moment and shearing force diagram for the bar. Details of the use of the diagram for this purpose will be found in the original paper by Howard (2) or in certain books dealing with the theory of structures (3). It may be emphasised here once more that the present use of the diagram does not necessitate accurate drawing; it is only used to enable geometrical relations to be obtained for the determination of formulas of a general nature and sketches only are needed. A number of cases will now be dealt with which will indicate the method.

Case 1. *The pin-jointed strut with an eccentric thrust.*

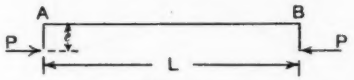

FIG. 3.

The strut AB, shown in Fig. 3, carries a thrust P which acts at a distance e from the axis of the bar. The end couples are therefore each equal to Pe, and, since there is no distributed load, $M=m$.

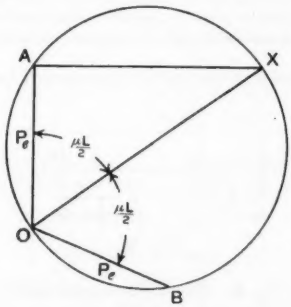

FIG. 4.

So, at $x=0$ and at $x=L$ the value of m is Pe, and the polar diagram is shown in Fig. 4 where $OA=OB=Pe$, and angles AOX and XOB are each equal to $\mu L/2$.

Then, $$M_{max}=m_{max}=Pe \sec \mu L/2.$$

Also, from equation (1),

$$M_{max}=-Py_{max}+Pe,$$

and so at the centre of the bar $y_{max}=e(1-\sec \mu L/2)$.

Case 2. Strut with an axial thrust and a uniformly distributed load.

This case is shown in Fig. 1, but the end couples are both made zero.

From equation (3), $m = M - w/\mu^2$.

At A and B, since there are no applied end couples, $m = -w/\mu^2$, so OA and OB in Fig. 5 are set off in the negative direction to represent w/μ^2. The m-circle is then drawn through O, A and B, and m_{max} is seen to be $-OX$ or $-w/\mu^2 \sec \mu L/2$.

Hence, $$M_{max} = m_{max} + w/\mu^2 = -w/\mu^2 \sec \mu L/2 + w/\mu^2.$$

or $$M_{max} = (1 - \sec \mu L/2)\, w/\mu^2,$$

which is negative, as we should expect.

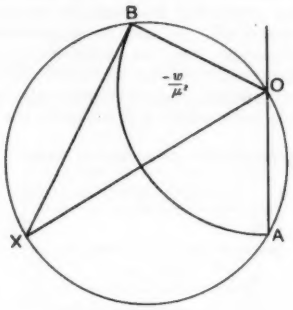

FIG. 5.

If in Fig. 5 we draw a circle of radius w/μ^2 with centre at O, the intercepts between this circle and the m-circle are the values of the bending moments at all points on the bar and we have a complete diagram.

Case 3. Strut with an axial thrust and a concentrated load.

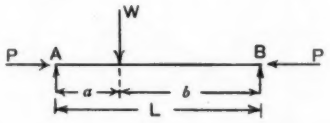

FIG. 6.

This case is shown in Fig. 6. The strut carries an axial thrust P as before and a single lateral load W at a distance a from A and b from B. There are no end couples. The values of m will now be represented by two circles, since the constants of integration will be different in the two sections of the bar into which the load divides it.

Also, since there is no distributed load, $M = m$.

To draw the diagram, which is shown in Fig. 7, make $ZOK = \mu L$ and draw DOY at an angle μa to ZO to represent the position of the load. The two m-circles will give values in the sectors ZOD and DOK respectively, and since the values of m must be the same from both circles at the load point, these circles must intersect on DOY.

Also, since there are no end couples, the circles must not cut the lines ZOZ' and KOK', so these lines will be tangents to the appropriate circles at O. The centre of the circle for sector ZOD will therefore lie on OX_1, which is perpendicular to ZO, and the centre of the other circle will lie on OX_2, which is perpendicular to KO. The diagram is therefore as shown in the figure. The maximum value of m and therefore of M occurs at the load point and is given by OY.

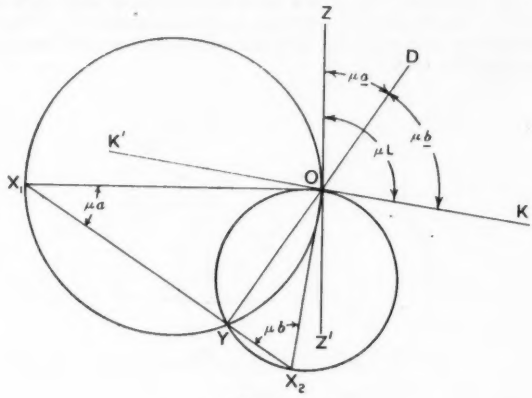

FIG. 7.

Since there is no change of slope in passing through the load point, it follows from equation (5) that the change of real shearing force in passing this point is the same as the change in apparent shearing force, or $S_R - S_L = S_R' - S_L' = W$, where S_R and S_L denote the values of the shearing forces to the right and left of the load point.

But $$S_L = -\mu X_1 Y \text{ and } S_R = \mu X_2 Y,$$

so $$X_2Y + X_1Y = X_1X_2 = W/\mu.$$

Then in the diagram, since OX_1 and X_1Y are respectively perpendicular to $0Z$ and OY, the angle $OX_1Y = \mu a$, and similarly the angle $OX_2Y = \mu b$.

Hence, $$X_1OX_2 = \pi - \mu L.$$

Now, $$OY = OX_1 \sin \mu a.$$

Also, $$OX_1/\sin \mu b = X_1X_2/\sin \mu L,$$

and so, $$OX_1 = X_1X_2 \sin \mu b/\sin \mu L = W \sin \mu b/\mu \sin \mu L.$$

Therefore,

$$M_{\max} = m_{\max} = -OY = -\frac{W}{\mu}\frac{\sin \mu a \sin \mu b}{\sin \mu L}.$$

If the load is central, $a = b = L/2$ and

$$M_{\max} = -(W/2\mu) \tan \mu L/2,$$

which is a well-known result.

Case 4. *Encastré strut with axial thrust and uniformly distributed load.*

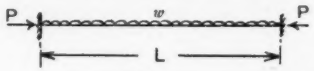

FIG. 8.

In this case, shown in Fig. 8, the strut is encastré at both ends and carries a uniform load of intensity w in addition to its axial thrust P. The end couples are now unknown and must be determined, as well as the bending moment at the centre of the bar.

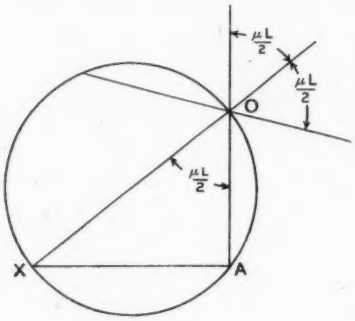

FIG. 9.

From equation (5), since the slope of the bar at the end is zero, the true shearing force is the same as the apparent shearing force, *i.e.* $\frac{1}{2}wL$. The diagram is as shown in Fig. 9, where AX is now known to be $wL/2\mu$.

Hence, $$m_{\max} = -OX = -(wL/2\mu)\operatorname{cosec}\mu L/2,$$

and $$M_{\max} = m_{\max} + w/\mu^2 = -\frac{wL}{2\mu}\operatorname{cosec}\frac{\mu L}{2} + \frac{w}{\mu^2}.$$

The value of m at the support is

$$-OA = -AX\cot\tfrac{1}{2}\mu L = -(wL/2\mu)\cot\tfrac{1}{2}\mu L.$$

The end fixing moment is therefore

$$M_A = -(wL/2\mu)\cot\tfrac{1}{2}\mu L + w/\mu^2.$$

Case 5. *Encastré strut with axial thrust and central load.*

This case of loading is shown in Fig. 10. The m-diagram will consist of two equal circles intersecting at the point Y in Fig. 11. As in the preceding case, we do not know the value of the end fixing moments, but the real and apparent shearing forces at the supports are both $\frac{1}{2}W$. So $AX = W/2\mu$. By the same reasoning, the true shearing force to the left of the load is also $\frac{1}{2}W$, and so $XY = W/2\mu$. The triangles OAX and OYX are therefore similar, and

$$AO = OY = (W/2\mu)\tan\tfrac{1}{2}\mu L.$$

Since $m = M$, this is the value both of the bending moment at the centre and at the supports, the former being negative and the latter positive.

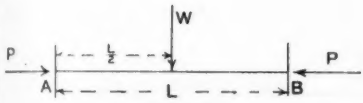

FIG. 10.

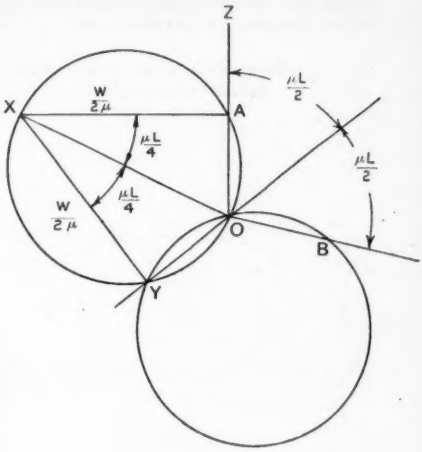

FIG. 11.

Case 6. *Pin-jointed strut with specified end couples.*

FIG. 12.

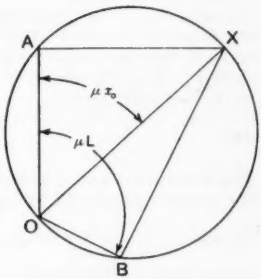

FIG. 13.

In this case, shown in Fig. 12, the strut carries an axial thrust P and specified end couples M_A and M_B. The m-circle is shown in Fig. 13. OX is the maximum value of m and therefore of M.

From the diagram,

$$OX = M_A \sec \mu x_0 = M_B \sec (\mu L - \mu x_0),$$

where $\mu x_0 = \epsilon$ is the position of maximum bending moment in the bar.

This gives $\tan \mu x_0 = (M_B/M_A) \operatorname{cosec} \mu L - \cot \mu L,$

Then, $OX^2 = M^2_A (1 + \tan^2 \mu x_0),$

and substituting the value of $\tan \mu x_0$, already found, we have

$$m_{max} = M_{max} = \sqrt{\{(M^2_A + M_B{}^2) \operatorname{cosec}^2 \mu L - 2M_A M_B \operatorname{cosec} \mu L \cot \mu L\}}.$$

If the end couples are equal this reduces to the same result as found in Case 1, where the end couples were each equal to Pe.

Case 7. Strut with a uniformly distributed load, pin-jointed at one end and encastré at the other.

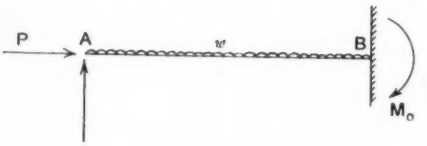

FIG. 14.

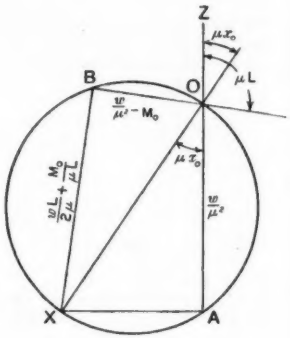

FIG. 15.

This case is shown in Fig. 14. If M_0 is the value of the unknown fixing moment at B, the apparent shearing force at B is $\frac{1}{2}wL + M_0/L$, and, since the slope of the beam there is zero, it is also the true shearing force. At the ends of the bar, $m_A = -w/\mu^2$ and $m_B = -(w/\mu^2 - M_0)$, so the m-circle is as shown in Fig. 15, where $OA = w/\mu^2$ and $OB = w/\mu^2 - M_0$, both being set out in the negative sector.

Also $$BX = wL/2\mu + M_0/\mu L.$$

The maximum value of m in the member is given by $-OX$ and it occurs at the point represented by the angle μx_0 on the diagram.

Then $$OX = (w/\mu^2 - M_0)\sec(\mu L - \mu x_0) = (w/\mu^2)\sec \mu x_0,$$

from which $$\tan \mu x_0 = (1 - \mu^2 M_0/w - \cos \mu L)\operatorname{cosec} \mu L.$$

Also, $$OX = (wL/2\mu + M_0/\mu L)\operatorname{cosec}(\mu L - \mu x_0) = (w/\mu^2)\sec \mu x_0,$$

which gives $$\tan \mu x_0 = \tan \mu L - (\mu^2/w)(wL/2\mu + M_0/\mu L)\sec \mu L.$$

The equation of the two values of $\tan \mu x_0$ gives

$$M_0 = \frac{wL}{\mu}\left(\frac{\cos \mu L - 1 + \frac{1}{2}\mu L \sin \mu L}{\mu L \cos \mu L - \sin \mu L}\right),$$

and then $$\tan \mu x_0 = \frac{2\mu L \sin \mu L + 2\cos \mu L - 2 - \mu^2 L^2}{2(\mu L \cos \mu L - \sin \mu L)}.$$

The maximum bending moment in the strut is

$$M_{\max} = -OX + w/\mu^2 = -w/\mu^2(\sec \mu x_0 - 1).$$

Case 8. *Strut with an axial thrust and a concentrated load, with one end pin-jointed and the other encastré.*

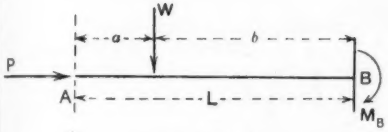

FIG. 16.

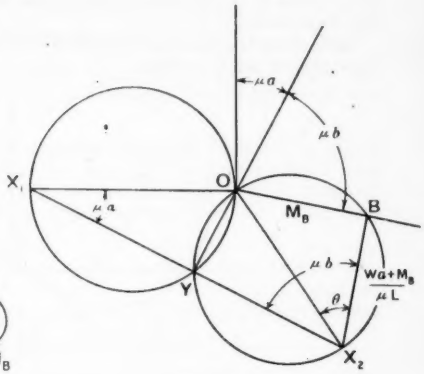

FIG. 17.

This case is shown in Fig. 16. The value of m at A is zero, and at B it is M_B, which has to be determined. The real and apparent shearing forces at B are the same since B is encastré, and so, in Fig. 17,

$$OB = M_B \text{ and } BX_2 = (Wa + M_B)/\mu L.$$

Let angle $BX_2O = \theta$.

Then
$$\left.\begin{aligned} OX_2 &= M_B \operatorname{cosec}\theta = (Wa + M_B)\sec\theta/\mu L \\ &= OY \operatorname{cosec}(\mu b - \theta) = X_1Y \tan \mu a \operatorname{cosec}(\mu b - \theta) \\ &= X_2Y \sec(\mu b - \theta). \end{aligned}\right\} \quad \text{.....(1)}$$

Also
$$X_1Y + X_2Y = W/\mu. \quad \text{.....(2)}$$

Substituting from (1) in (2), eliminating θ, we have

$$M_B = \frac{W}{\mu}\left(\frac{\mu a \sin \mu L - \mu L \sin \mu a}{\mu L \cos \mu L - \sin \mu L}\right).$$

And the bending moment under the load is then

$$OY = \{(Wa + M_B)\sin \mu b\}/\mu L - M_B \cos \mu b.$$

Case 9. *Strut with an axial thrust and a concentrated load with both ends encastré.*

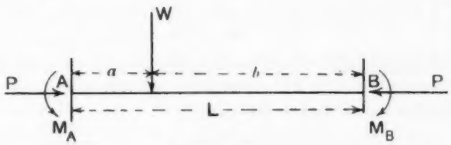

FIG. 18.

This case is shown in Fig. 18. The end fixing moments M_A and M_B are unknown, and these, as well as that under the load, have to be determined. Since there is no distributed load, the values of m at A and B are M_A and M_B respectively.

Also, since the ends are encastré, the real and apparent shearing forces at these points are the same.

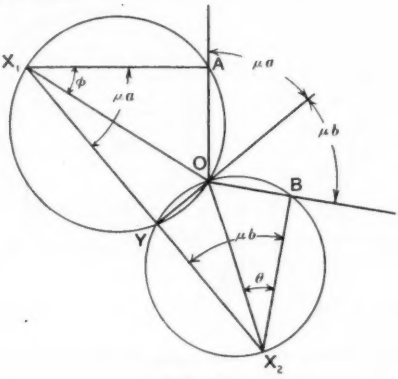

FIG. 19.

Hence in Fig. 19, $OA = M_A$, $OB = M_B$,

$$AX_1 = (Wb + M_A - M_B)/\mu L \text{ and } AX_2 = (Wa - M_A + M_B)/\mu L.$$

Let angle $AX_1O = \phi$ and $BX_2O = \theta$.

Then
$$\left.\begin{aligned} OX_1 &= M_A \operatorname{cosec}\phi = \{(Wb + M_A - M_B)\sec\phi\}/\mu L \\ &= OY \operatorname{cosec}(\mu a - \phi) = X_1Y \sec(\mu a - \phi) \end{aligned}\right\} \quad \ldots\ldots(1)$$

$$\left.\begin{aligned} OX_2 &= M_B \operatorname{cosec}\theta = \{(Wa - M_A + M_B)\sec\theta\}/\mu L \\ &= OY \operatorname{cosec}(\mu b - \theta) = X_2Y \sec(\mu b - \theta) \end{aligned}\right\} \quad \ldots\ldots(2)$$

and
$$X_1Y + X_2Y = W/\mu. \quad \ldots\ldots(3)$$

Equating the values of OY obtained from groups (1) and (2), we find, on eliminating ϕ and θ,

$$M_A(\sin\mu a + \sin\mu b - \mu L\cos\mu a) - M_B(\sin\mu a + \sin\mu b - \mu L\cos\mu b) = W(a\sin\mu b - b\sin\mu a). \quad \ldots\ldots(4)$$

Using (3), (2) and (1) we obtain

$$M_A(\cos\mu a - \cos\mu b + \mu L\sin\mu a) - M_B(\cos\mu a - \cos\mu b - \mu L\sin\mu b) = W(L - a\cos\mu b - b\cos\mu a). \quad \ldots\ldots(5)$$

Hence,

$$M_A = \frac{W}{\mu}\left[\frac{\sin\mu a + \sin\mu b - \sin\mu L + \mu a + \mu b\cos\mu L - \mu L\cos\mu b}{2(1-\cos\mu L) - \mu L\sin\mu L}\right]$$

$$M_B = \frac{W}{\mu}\left[\frac{\sin\mu b + \sin\mu a - \sin\mu L + \mu b + \mu a\cos\mu L - \mu L\cos\mu a}{2(1-\cos\mu L) - \mu L\sin\mu L}\right].$$

Then the bending moment under the load is

$$OY = M_A\sin(\mu a - \phi)\operatorname{cosec}\phi$$

or
$$OY = \{(Wb + M_A - M_B)\sin\mu a\}/\mu L - M_A\cos\mu a.$$

The author wishes to thank Miss L. Chitty and Dr. W. G. Bickley for their solution of Cases 7 and 8 and for suggestions and help in this paper.

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2. "The Graphical and Analytical Determination of Stresses in Single Span and Continuous Beams under End Compression and Lateral Load with Variation in Shear, Distributed Load and Moment of Inertia."—H. B. Howard. *Reports and Memoranda of Aeronautical Research Committee,* No. 1233. His Majesty's Stationery Office.

3. For example, *The Analysis of Engineering Structures.*—Pippard and Baker. Pp. 148-153. Arnold.

A. J. S. P.

GLEANINGS FAR AND NEAR.

1399. I (Archdeacon Wilson) remember his being strongly attracted by a curious chance problem, of which I published a solution in the Philosophical Magazine: "If four points are taken at random on an infinite plane, to find the probability that one will fall inside the triangle formed by the other three." Several solutions giving different results had been given, and he could not satisfy himself with any treatment of the question. (p. 60.)

1400. Another peculiarity of his mathematical powers may be mentioned—his power of *seeing, without counting,* comparatively large numbers. Everybody *sees* three, not as two and one, but as three; most people see four; some see five, and even six; more frequently, careful observation of themselves will show that there is a rapid addition of three and two, or three and three. But Dr. Temple certainly *saw* higher numbers. I tested him quite suddenly more than once—How many sheep in that field? Instantly the answer came *nine*; once he *saw* thirteen. . . . (p. 61.)

1401. When Bishop of Exeter . . . he discovered . . . that boys were in the habit of coming to be confirmed a second time. The clergyman would bring the list to the vestry with, say, 40 names on it. "But there were 43," said the Bishop. "No, my Lord, only 40," was the reply. "But I saw 43." "Did your Lordship count them?" "No, I *saw* them," and it turned out that he was right. (p. 61.)

1402. One of his recreations in his latest years was an endeavour to find some law governing the occurrence of a prime, and to find the greatest series of consecutive numbers without a prime. (p. 61.)

1403. His son was instructed to design and provide working drawings for the junction between two lengths of a tunnel, of which the sections were different. The problem is highly technical, and somewhat novel. The brick lining must give the same resisting power. The transition is effected by the use of stones of a truncated wedge shape. . . . The Archbishop mastered it, and on the following day said that at one point . . . the design was in error. He gave no reason but . . . he was found to be right in his remark. (p. 61.)

1404. I (Archdeacon Wilson) well remember when I was reading Herbert Spencer's First Principles, and distrusted my own conclusion that the physics and mathematics of the first book were hopelessly confused, that I talked it over with Dr. Temple. He had come to exactly the same conclusion. (p. 62.)

Gleanings 1399-1404 from *Memoirs of Archbishop Frederick Temple,* I. [Per Mr. A. F. Mackenzie].

AMENDED LAWS OF INDICES.

By W. F. Dunton.

Writers on algebra have noticed that the usual laws of indices lead to self-contradictions (Chrystal called them "paradoxes"), no matter whether the meaning of a fractional index is defined by

$$z^{p/q} = (z^p)^{1/q} \quad \text{.....(1)}$$

or by

$$z^{p/q} = (z^{1/q})^p ; \quad \text{.....(2)}$$

but it does not appear that anyone has yet shown how the laws can be modified in a way that makes them self-consistent. In this note, the necessary amendments are stated for each of these definitions, and are shown to be very much simpler for definition (1). This is fortunate; for that definition appears to be the generally accepted one, though definition (2) is adopted by Hardy and some others.

Throughout this note, z represents a number of the form $m(\cos\theta + i\sin\theta)$, where the modulus m may be any arithmetical number, and the amplitude θ (otherwise called the argument) may be positive, negative or zero. The number becomes m when $\theta = 0$, becomes $-m$ when $\theta = \pi$, and may therefore be any positive, negative or complex number. The letters p, q, r and s in the indices are all supposed to be positive integers; but the subject of negative indices is referred to in the final paragraph.

When the matter is dealt with on the basis of the well-known fact that $z^{1/q}$ has q values and that the amplitudes of these are

$$\theta/q + 2\pi k/q \text{ with } k = 0, 1, 2, \ldots, q-1, \quad \text{.....(3)}$$

the usual laws of indices are found to require the amendments indicated below. There is never any doubt with regard to the modulus, and the investigation can therefore be simplified by considering merely the amplitudes.

The usual first law $z^{p/q}\, z^{r/s} = z^{p/q + r/s} = z^{\frac{ps+qr}{qs}}$, ...(I)

should be $z^{p/q}\, z^{r/s} = z^{\frac{(ps+qr)/a}{qs/a}}$ .....(I′)

when definition (1) is adopted, a being the greatest common factor of q and s, and should be

$$z^{p/q}\, z^{r/s} = \{z^{(ps+qr)/cdf}\}^{\frac{1}{qs/cdf}} \quad \text{.....(I″)}$$

when definition (2) is adopted, c being the greatest common factor of p and q, d being the greatest common factor of r and s, and f being the greatest common factor of q/c and s/d.

The usual second law $(z^{p/q})^{r/s} = z^{pr/qs}$ .....(II)

should be $(z^{p/q})^{r/s} = z^{\frac{pr/b}{qs/b}}$ .....(II′)

when definition (1) is adopted, b being the greatest common factor of q and r, and should be

$$(z^{p/q})^{r/s} = \{z^{pr/cdg}\}^{\frac{1}{qs/cdg}} \quad \text{.....(II″)}$$

when definition (2) is adopted, c being the greatest common factor of p and q, d being the greatest common factor of r and s, and g being the greatest common factor of q/c and r/d.

The usual third law $y^{p/q}\, z^{p/q} = (yz)^{p/q}$, ...(III)

holds good without any amendment, no matter whether definition (1) or definition (2) is adopted.

The method of proof is best explained by the following applications to laws (II′) and (II″). Readers whose previous knowledge of the subject enables them to follow these proofs, will have no difficulty in applying the same method to the other cases, which are less complicated.

A reference to (3) will show that the amplitudes of $(z^{p/q})^{r/s}$ are, when definition (1) is adopted,

$$\{\theta pr/q + 2\pi kr/q\}/s + 2\pi t/s \text{ with } t = 0, 1, 2, \ldots, s-1.$$

But the total number of amplitudes may be less than qs; for if q and r have a greatest common factor b, so that $q = bw$ and $r = bx$, then $kr/q = kbx/bw$, which becomes an integer as soon as k becomes w. With $k = w$, then, the amplitudes will be essentially the same as with $k = 0$; with $k = w+1$, they will be essentially the same as with $k = 1$; and so on: that is to say, the expressions for these amplitudes will differ merely by 2π or by some multiple of 2π, and they will therefore indicate the same values of z. The number of values of z will then be only qs/b. It cannot be less than qs/b; for the coefficient of 2π is $(kx/w + t)/s$, with k less than w, so that kx/w cannot be a positive integer, whereas t must be either a positive integer or zero; and it follows that no change in k can compensate for any change in t, so that the w values of k and the s values of t must give ws different values of $(kx/w + t)/s$. This proves law (II′). It shows, for example, that $(z^{2/3})^{9/4}$ has the 4 values represented by $z^{6/4}$, whereas law (II) says that this expression is equivalent to $z^{18/12}$, which has 12 values when definition (1) is adopted. Even if law (II) is thought to mean that the index 18/12 should be reduced to its lowest terms, it does not give the correct result; for the index then becomes 3/2, and the number of values of the power is thus wrongly given as 2. Serious mistakes have been made, by engineers and mathematical physicists, through reducing fractional indices to their lowest terms; notably through applying law (II) to $(\cos^2\theta)^{1/2}$ and inferring that this is simply $\cos\theta$.

When definition (2) is adopted, and when p'/q' represents p/q in its lowest terms, we have

$$z^{p/q} = (z^{1/q})^p = (z^{1/q'})^{p'} = (z^{p'})^{1/q'}.$$

This is made clear by the above proof of law (II′), in which it was shown that $(z^{1/q})^p$ has q' values. Investigations with definition (2) can therefore be facilitated by changing the indices p/q and r/s to the p'/q' and r'/s' that represent these fractions in their lowest terms, and then proceeding as in the case of definition (1). The number of values of $(z^{p/q})^{r/s}$ with definition (2) is thus quickly found to be the same as the number of values of $(z^{p'/q'})^{r'/s'}$ with definition (1), namely $q's'/g$, where g is the greatest common factor, is q' and r'. But, with definition (2), these values are indicated by $(z^{p'r'/g})^{\frac{1}{q's'/g}}$, which is not equivalent to $z^{\frac{p'r'/g}{q's'/g}}$ (except when this last index happens to be a fraction in its lowest terms). Law (II″), in which q/c and s/d are equal to q' and s' respectively, is therefore now seen to be correct. It shows, for example, that $(z^{6/4})^{2/12}$ has the 12 values represented by $(z^3)^{1/12}$, whereas law (II), with definition (2), says that this expression has only the 4 values represented by $z^{12/48} = (z^{1/48})^{12} = z^{1/4}$.

With regard to negative indices, it will be sufficient to point out that

$$z^{p/(-q)} = z^{-(p)/q} = z^{-(p/q)} = (1/z)^{p/q}.$$

The correctness of these equations will be seen from the way in which the subject of negative and fractional indices is dealt with in books on elementary algebra.

W. F. DUNTON.

MATHEMATICAL NOTES.

1594. *On Note* 1551 (XXV, p. 249).

The "accepted rule" for the sign of the perpendicular from a point to a line is stated and criticised on p. 249 of the October *Gazette*. It may be doubted whether the rule has in fact ever been generally accepted. If it has, there are valid reasons for discarding it: amongst them the argument that the origin has nothing to do with the case.

The really important fact is that ax_1+by_1+c and ax_2+by_2+c have the same or opposite signs according as (x_1, y_1) and (x_2, y_2) are on the same or opposite sides of the line $ax+by+c=0$. This follows by substitution of the coordinates $(k_1x_1+k_2x_2)/(k_1+k_2)$, $(k_1y_1+k_2y_2)/(k_1+k_2)$ into the equation of the line, and it answers the usual questions of elementary Cartesian geometry dealing with perpendiculars and angle-bisectors.

Apparently it has become customary in certain examinations to ask questions about bisectors of obtuse angles and of angles in which the origin lies. The reaction among writers of text-books and short-sighted schoolmasters has been to design a rule to ensure that no schoolboy will ever make a mistake in a question of this kind. This idea is founded upon a gross underestimate of the ingenuity of the young. The next development was a discovery by examiners of a flaw in the rule: they found that by making a line pass through the origin they could set a question to make the candidate think; and if they were not frustrated by the moderators, they doubtless succeeded in defeating most of the candidates. Thus they were able to penalise candidates who had been foolish enough to learn a foolish rule from a foolish master. But now Mr. C. Black has come to the rescue with an infallible device for lifting the unfortunate student out of the frying-pan into the fire.

To illustrate the kind of questions that arise we may take the "important example" chosen by a well-known writer (Charles Smith, in *Conics*, p. 37); this is to find the in-centre I of the triangle $A(1, 2)$, $B(25, 8)$, $C(9, 21)$. There are two simple methods available.

(i) Since $BC : CA : AB = 5 : 5 : 6$, $I(x, y)$ is the centroid of 5, 5, 6 at A, B, C, and so

$$x=(5\,.\,1+5\,.\,25+6\,.\,9)/16=11\tfrac{1}{2},$$

$$y=(5\,.\,2+5\,.\,8+6\,.\,21)/16=11.$$

(ii) The equations of BC, CA, AB are

$$13x+16y-453=0,\ 19x-8y-3=0,\ x-4y+7=0;$$

and the distances of the in-centre from them are

$$-\frac{13x+16y-453}{5\sqrt{17}},\quad +\frac{19x-8y-3}{5\sqrt{17}},\quad -\frac{x-4y+7}{\sqrt{17}},$$

where the signs are found by some such methods as these:

(*a*) the first sign is $-$ because I is on the same side of BC as A, and $13x+16y-453$ is negative when x, y are the coordinates of A ;

(*b*) the second sign is $+$ because I is on the same side of CA as a point a long way off on the positive x-axis, and $19x-8y-3$ is positive when (x, y) is such a point ;

(*c*) the third sign is $-$ because I is on the opposite side of AB from the origin, and $0-4\,.\,0+7$ is positive.

Hence I is given by

$$-13x-16y+453=19x-8y-3=5(-x+4y-7).$$

There is no obvious reason for deciding always to use one particular method such as (*a*) or (*b*) or (*c*), or for deciding always to write $ax+by+c=0$ with c positive or with b negative. Method (*c*) is the traditional one ; this method will not work if AB is a line through O ; nor would Mr. Black's method work for lines parallel to OY if he had not been careful to deal with the case of $b=0$. But surely the limit of absurdity is reached when a rule is given to find the perpendicular distance of (x_1, y_1) from $x=a$. There can be no case in which this distance is not immediately obvious, as also the side of the line on which the point lies. To decide the sign of the perpendicular is not to answer a question of geometry, but only to solve an unnecessary problem that one has set oneself by inventing a convention. Similar remarks apply to more complicated questions, except that the answers only become obvious after a consideration of the result about ax_1+by_1+c and ax_2+by_2+c.

The fact that there is a sign convention for distances along OX is no reason why there " must " be one for other lines. And is there in fact any such convention? Is it not a convention about coordinates rather than distances?

When a convention is proposed, the minimum test to be satisfied before it can be accepted is surely that there shall be at least one problem that is not made more complicated when the convention is used. A. R.

1595. *On Note* 1551 (XXV, p. 249).

There are occasions when the deferential " it seems to me " seems to me inappropriate, and I assert obstinately that the formulation of rules for attaching a definite sign to the perpendicular from (x', y') to $ax+by+c=0$ is misapplied labour. The purpose of analytical geometry is to prove properties of figures in general by means of a special scaffolding ; unless all trace of the scaffolding ultimately disappears, the property established is not absolute but relative in some way to the frame of reference. It follows that there can be nothing fundamental about a sign convention which produces results dependent on the axes ; such a convention is part of the scaffolding, and the only question is whether it is useful.

In problems for which a sign has been thought to be wanted for a perpendicular, the distinction to be drawn is always essentially between two points on the same side of a line and two points on opposite sides of the line, and this distinction is effected immediately by a comparison of the signs of $ax'+by'+c$ and $ax''+by''+c$. The points for which $ax+by+c$ is positive compose one region of the plane, the points for which $ax+by+c$ is negative compose another region, and these regions are separated by the aggregate of points for which $ax+by+c$ is zero, that is, by the line itself; the discrimination between the two regions is not assisted in the least by a convention attaching the positive sign to one of them.

The argument that "just as we have a convention of signs for distances along OX or OY, so we must have a convention of signs for any straight line in the plane" is arrant nonsense. We have also a convention of measuring distances along OX and OY from the specified origin O, but we are not therefore under compulsion to nominate a unique origin of measurement on an arbitrary line. Directions and origin alike are wanted on OX and OY for the Cartesian use of these lines, and have no validity as precedents for lines that are not being put to that use.

There is, indeed, a plane geometry in which all the lines, and all the circles too, have direction, and a fascinating subject it is: there is only one "cycle" which touches three "rays", two cycles have only one centre of similarity, and the locus of the centre of a variable cycle which touches two fixed cycles is a single conic easily identified. But we do not enter this geometry by assigning a unique direction to each undirected line and to each unoriented circle; in the "geometry of direction" each line is the axis of two rays, each circle is a groove occupied by two cycles. E. H. N.

1596. *On Note* 1522.

It is more easily proved by coordinate geometry that any circle on a focal chord of a parabola as diameter touches the directrix and a fixed circle with its centre on the axis. Take the parabola as $y^2=4ax$ and the ends of the chord as $(as^2, 2as)$, $(at^2, 2at)$. The equation of the chord is

$$(s+t)y=2(x+ast),$$

and since it goes through the focus $(a, 0)$ we have $st=-1$. The equation of the circle with the chord as diameter is

$$x^2+y^2-a(s^2+t^2)x-2a(s+t)y+a^2st(4+st)=0,$$

and so, putting $s+t=\rho$, $st=-1$, this becomes

$$x^2+y^2-a(\rho^2+2)x-2a\rho y-3a^2=0. \quad \text{.....(i)}$$

The envelope is found as the condition that the equation in ρ,

$$ax\rho^2+2ay\rho-(x^2+y^2-2ax-3a^2)=0,$$

shall have equal roots.

This condition is $a^2y^2+ax(x^2+y^2-2ax-3a^2)=0$,

or $$(x+a)(x^2+y^2-3ax)=0.$$

Thus the envelope is the directrix and the circle $x^2+y^2-3ax=0$, whose centre is on the axis. It is readily worked out that $x+a=0$ touches the variable circle (i) at the point $(-a, a\rho)$, and that

$$x^2+y^2-3ax=0$$

touches it at $\{3a/(1+\rho^2), -3a\rho/(1+\rho^2)\}$.

These two points obviously lie on the line $y=-\rho x$, a line through the vertex. It may be proved that for circles on chords of the parabola as diameters, all the chords being concurrent in a point on the axis, the envelope will only become a straight line and a circle when the fixed point is the focus. LAWRENCE CRAWFORD.

1597. *On Note* 1550: *a simple approach to Pythagoras' theorem.*

If Pythagoras is to come after similar triangles, why drag in the properties of the circle also? Euclid has done it for us, in more general form, in VI, 31. It can be put very shortly, thus, in this particular case:

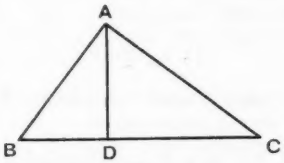


FIG.

From the similar triangles CBA, ABD,

$$CB : BA = AB : BD,$$

or $$CB \cdot BD = AB^2.$$

From the similar triangles CBA, CAD,

$$BC : CA = CA : CD,$$

or $$BC \cdot CD = CA^2;$$

hence $$BC(BD+DC) = AB^2+CA^2,$$

that is, $$BC^2 = AB^2+CA^2.$$

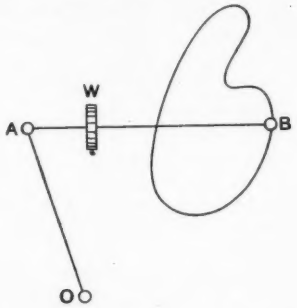
No originality is claimed for this simple proof. G. J. LIDSTONE.

1598. *Amsler's Planimeter.*

Amsler's planimeter consists of two rods OA and AB, jointed at A. B makes a circuit of a closed curve drawn on a piece of paper, while O remains fixed to the paper. AB carries a wheel W whose axis is parallel to AB, and which moves over the paper partly by sliding and partly by rotating. We have to prove

(1) That the distance l through which W rotates as B makes the circuit is independent of the position of W on AB;

(2) That the area of the closed curve is $AB \times l$.



These results are easily proved vectorially by using the fact that if $\mathbf{r}$ is the position vector of a point on a plane curve of area S, taken relative to a point in the plane of the curve, and $\mathbf{k}$ is a unit normal to the plane, then

$$\tfrac{1}{2}\int_c \mathbf{r} \wedge d\mathbf{r} = \mathbf{k}S,$$

the integral being taken round the curve. In the following, all integrals refer to a circuit of the curve.

Let $\overrightarrow{OA}$, $\overrightarrow{AB}$ and $\overrightarrow{AW}$ be $\mathbf{a}$, $\mathbf{b}$ and $n\mathbf{b}$ respectively.

Then
$$\mathbf{k}bl = \int \mathbf{b} \wedge d(\mathbf{a} + n\mathbf{b})$$
$$= \int \mathbf{b} \wedge d\mathbf{a},$$

since $\int \mathbf{b} \wedge d\mathbf{b} = 0$, for if we make $\mathbf{b}$ a position vector it traces out zero area. This shows that $\mathbf{k}bl$ is independent of n and the result (1) follows.

Again, for the closed curve whose area is required,

$$\mathbf{k}S = \tfrac{1}{2}\int (\mathbf{a} + \mathbf{b}) \wedge d(\mathbf{a} + \mathbf{b})$$
$$= \tfrac{1}{2}\int (\mathbf{b} \wedge d\mathbf{a} + \mathbf{a} \wedge d\mathbf{b}),$$

since $\mathbf{a}$ and $\mathbf{b}$ both trace out zero areas,

$$= \tfrac{1}{2}\Big[\mathbf{a} \wedge \mathbf{b}\Big]_c + \int \mathbf{b} \wedge d\mathbf{a},$$

for
$$\mathbf{a} \wedge d\mathbf{b} = d(\mathbf{a} \wedge \mathbf{b}) + \mathbf{b} \wedge d\mathbf{a}.$$

The first term vanishes since **a** and **b** return to their original positions. Hence

$$\mathbf{k}S = \int \mathbf{b} \wedge d\mathbf{a} = \mathbf{k}bl,$$

thus proving (2). H. F. BROWNE.

1599. *Pascal's Theorem.*

If the proof by means of a cubic curve is to be used for beginners, it is desirable to put it into an elementary form.

Thus the first of the two proofs in Salmon's *Higher Plane Curves* is unsuitable because it requires the preliminary theorem that any cubic through the points of intersection of two cubics $U=0$ and $V=0$ has an equation $U=kV$. Also the proof in Note 1547 of the *Gazette* assumes that any cubic through the points of intersection of $S=0$ and $\alpha\beta\gamma=0$ has an equation $LS=\alpha\beta\gamma$.

One would not wish to raise these difficulties with beginners.

Salmon's second proof (*l.c.*, p. 17 in the second edition) is essentially elementary. The constant k can be chosen so that the cubic $\alpha\beta\gamma=k\alpha'\beta'\gamma'$ passes through a seventh point of the conic; it follows that the conic is part of the cubic. The proof of the preliminary theorem used here is immediate: substitution from parametric equations of the conic into the equation of the cubic gives a relation of degree 6 in the parameter; since this relation is true for 7 values of t, it is true for all values. A. R.

1600. *Analytical Proof of Brianchon's Theorem.* (Cf. Note 1548, XXV, p. 248.)

Retain the wording of Note 1548, substituting tangential coordinates for trilinear.

If abc is a triangle of reference and p, q, r are any three lines in the plane, the points cq, br are $l/l_q=m/m_q$, $l/l_r=n/n_r$ and their join is $(l_q l_r, m_q l_r, l_q n_r)$.

Brianchon's Theorem is the identity

$$\begin{vmatrix} l_q l_r, & m_q l_r, & l_q n_r \\ m_r l_p, & m_r m_p, & n_r m_p \\ l_p n_q, & n_p m_q, & n_p n_q \end{vmatrix} = \begin{vmatrix} m_p n_p, & n_p l_p, & l_p m_p \\ m_q n_q, & n_q l_q, & l_q m_q \\ m_r n_r, & n_r l_r, & l_r m_r \end{vmatrix}.$$

RAYMOND SMART.

1601. *The sign of a term in a determinant.*

The chief difficulty in teaching the theory of determinants, and a minor cause of trouble in their practical application, is the lack of a simple rule giving the sign of any term in the expansion of the determinant. Perhaps the following rule, which is a simple modification of one given by Aitken in his book *Determinants and Matrices*, may fill this want.

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

Let the determinant be set out in the usual way as a square matrix of elements. From each element which comes into the term in

question let a line be drawn upwards to the top of the matrix, and one horizontally to the left, as far as the edge of the matrix. Let the number of intersections of such horizontal and vertical lines be found. Then the sign of the term is + or − according as this number is even or odd.

Thus, if we take as an example the term $a_3b_4c_2d_1$ in the expansion of $\Sigma \pm a_1b_2c_3d_4$, we see from the diagram that the number of intersections of lines drawn in agreement with this rule is 5, and so the sign is −.

The same result will be obtained, of course, if all the vertical lines are drawn downwards, instead of upwards, and at the same time all the horizontal lines drawn to the right instead of the left.

C. A. B. SMITH.

1602. *On Note* 1433.

Further examples of the use of the notation $x_{\underline{p}} = x^{\underline{p}}/p!$ in bringing out the symmetry between the exponential series and the Gamma-integral, and between the binomial series and the Beta-integral, are the following:

$$(1)\quad \int_{x=0}^{X} e^{-x} dx_{\underline{R}} = \sum_{r=R}^{\infty} e^{-X} X_{\underline{r}};$$

$$(2)\quad \int_{x=0}^{X} (Y-x)_{\underline{S-R}} dx_{\underline{R}} = \sum_{r=R}^{S} (Y-X)_{\underline{S-r}} X_{\underline{r}}.$$

(1) gives the connection, of use in Statistics, between the "χ^2 distribution" and the "Poisson distribution" (with $X = \frac{1}{2}\chi^2$). (2) is equivalent to the similar connection between the "variance ratio distribution" and the "binomial distribution". (See R. A. Fisher and F. Yates, *Statistical Tables*, pp. 1, 2.) C. A. B. SMITH.

1603. *On the notation for a function.*

A cause of a certain amount of trouble with the present notation, $f(x)$, for a function of x, or for an operation on x, is that when it is part of a complicated expression a great number of brackets are often required to make it clear and unambiguous. Such brackets are often confusing to the eye.

In addition, the brackets are frequently omitted, but not in any systematic way. Thus dx^2 is used for $(dx)^2$, but $\cos^2 x$ for $(\cos x)^2$. $\mathrm{Cos}^{-1}x$ is used, however, for the inverse function, and not for the reciprocal.

For these reasons the following modification suggested itself to me as being a great help. In place of using both brackets in $f(x)$, only the right-hand one is kept. The left-hand bracket is omitted: in the old notation it never occurs apart from the letter f symbolising the function, and so is unnecessary.

Thus we may put $fx)$ for the function itself. The square of the function may then be written unambiguously as $fx)^2$, and the function of x^2 as $fx^2)$. The variable x may be replaced without difficulty by any other expression in this notation: thus we have

$fx+1$), distinguished from $fx)+1$, and the function f of the function g may be written fgx)). The inverse function, or operation, may be symbolised by $f^{-1}x$), and the repeated (iterated) function ffx)) by f^2x). Thus it is quite easy to compress such an expression as

$$1\Big/\left(\frac{d}{dx}\right)^2[f^{-1}(x^3)]^{\frac{1}{2}}$$

into $D^2f^{-1}x^3)^{\frac{1}{2}})^{-1}$.

A function of more than one variable may be written thus :

$$fx \mid y \mid z).$$

After having made use of the notation for some time, I have not come across any disadvantages other than unfamiliarity.

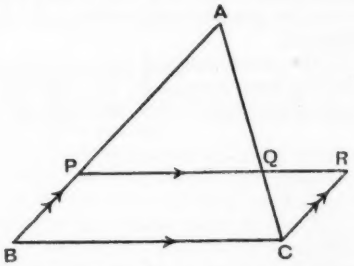
C. A. B. SMITH.

1604. *Similar triangles and proportional division.*

The introduction of similar triangles in stage *B* was recommended in the 1938 Geometry Report to come immediately after congruent triangles : " Having seen that the conditions for congruence are an outcome of the methods used for constructing triangles, it is easy to deal with similar triangles in the same way."

In several recent geometry textbooks this suggestion is adopted, though perhaps in a tentative manner. A corollary, however, which seems to have escaped notice, is that similarity of triangles comes *before*, and not *after*, the proportional division of the sides of a triangle by a line parallel to the base.

If this is agreed, we may give up the hitherto unending argument as to whether it is better to prove the proportional division theorem as a deduction from the equal intercept theorem (when it is easy to see but hard to write out) or to use equivalent triangles and triangles of equal altitudes which are proportional to their bases (when it is harder to see but easier to write out). For we may prove it by similar triangles as one of the " mid-point theorem " group.



Given PQ parallel to BC, draw CR parallel to BP to meet PQ produced at R. The triangles APQ, CRQ are equiangular and so, similar. Hence,

$$AP : RC = AQ : QC.$$

But RC, PB are opposite sides of a parallelogram.

Therefore $$AP : PB = AQ : QC.$$

Has anyone tried this proof? It is probably easier than the more direct proof using triangles APQ, ABC and 'dividendo'. It might be urged in its favour that riders on similar triangles are found to be easier than those on proportional division. C. O. TUCKEY.

1605. *The trigonometrical ratios.*

In textbooks on Trigonometry the discussion of the trigonometrical ratios of angles greater than 90° is invariably unsatisfactory and incomplete.

The first reference to the trigonometrical ratios, *sine*, *cosine*, etc., is in connection with right-angled triangles in which the ratios bear an obvious relation to the angle under consideration.

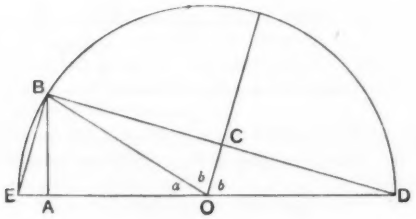
Later in the book it is explained that an angle should be regarded as the rotation of one line round another fixed line from a point in the latter, and that trigonometrical ratios for all such angles are obtainable from the right-angled triangle formed by dropping a perpendicular from a point in the rotating line on to the fixed line, and by assigning positive or negative values to the sides of the right-angled triangle so formed in accordance with a certain convention; the acute angle chosen to represent the degree of rotation being that between the rotating line and the fixed line.

Any intelligent student can hardly fail to find something lacking in the arbitrary definition for the trigonometrical ratios so derived. Why should the acute angle chosen to represent the degree of rotation be that between the final position of the rotating line and 180° and 360°?

What connection is there between the ratios of an angle less and of one greater than 90°?

Though no explanation is ever given in textbooks, it is easy to show that a geometrical connection exists and that the definition of the ratios for angles greater than 90° is not the whim of some ancient mathematician.

For instance, in the accompanying figure, the angle $(180 - a)$ can be regarded as having been formed by the rotation of OB from OD in two equal jumps of an angle b, each less than 90°.



Then $$\sin a = \frac{AB}{OB} = \frac{AB}{BD} \cdot \frac{BD}{OB} = \frac{OC}{OB} \cdot \frac{BD}{OB} = \frac{OC}{OB} \cdot \frac{2BC}{OB}$$

$$= 2 \sin b \cos b = \sin 2b.$$

Also $$\cos a = \frac{OA}{OB} = \frac{OB - EA}{OB} = 1 - \frac{EA}{OB} = 1 - \frac{EA}{AB} \cdot \frac{AB}{OB}$$

$$= 1 - \frac{OC}{CD} \cdot \frac{AB}{BD} \cdot \frac{BD}{OB} = 1 - \frac{OC}{CD} \cdot \frac{OC}{OB} \cdot \frac{2CD}{OB}$$

$$= 1 - 2\frac{OC^2}{OB^2} = 1 - 2 \cos^2 b$$

$$= -\cos 2b.$$

If the rotating line stops in the third quadrant it can be shown that $\sin a = -\sin 3b$, if in the fourth quadrant that $\sin a = -\sin 4b$, etc.

I suggest that it was from a consideration of the above that the values for ratios of angles greater than 90° were evolved and the apparently arbitrary definition laid down. A. H. B.

1606. *On Note* 1536.

With reference to Mr. T. H. Foster's interesting notes on Factorisation in the October *Gazette* (Note 1536, pp. 229-32), may I point out that the method used to factorise trinomials of the general form $ax^2 + bx + c$, where a is not unity, is certainly not new. For over thirty years I have taught this process and it may be found in Barnard and Child's *New Algebra*, the first edition of which was published in 1908. The method is also included in my *Mathematics for Technical Students*, Part II, pp. 280-1, and also in my recently published little book, *A Short Practical Mathematics*, p. 68.

Finally, in revising the late Frank Castle's *Elementary Practical Mathematics for Technical Students* in 1933, I introduced the method on pp. 89-90. All the books referred to are published by Messrs. Macmillan.

It is a pity, I agree, that the method is not to be found in many school algebras. F. G. W. Brown.

1607. *On Note* 1536.

Will Mr. Foster allow one less experienced than he to comment on his methods of factorisation?

In expressions of four terms the difficulty certainly is to prevent the weak child from pursuing too far the results of wrong pairing, but a cure for this is to insist on a check by multiplication of the final factors. The pupil may be disinclined to do this, but would he be more inclined to examine the terms as critically as Mr. Foster suggests? The relation between $(y - x)$ and $(x - y)$ should be established during the necessary preliminary of finding products.

In trinomial expressions such as $x^2 - 11x - 840$, the emphasis on the difference of 11, rather than the product of 840, is interesting, but that it makes it easier to find the required factors is open to

question. Personally, I find that most pupils readily appreciate that if their first choice, 10 and 84 say, gives too great a difference, they must seek a pair more nearly equal. Where the difference is small compared with the product, they soon see that an approximate square root of the product gives a good clue to the values of the factors, so that their first choice is often correct. In any case, it certainly can be said that each failure *does* give guidance for the next choice.

The method of reducing the trinomial ax^2+bx+c to an expression of four terms by considering the factors of (ac) with sum, or difference, b, is given in at least one well-known textbook, and gives no fresh difficulty to the pupil who has mastered the easier trinomial and four-term expressions. Another modern textbook shows how any trinomial may be expressed as the difference of two squares. Taking Mr. Foster's example,

$$\begin{aligned} 40x^2+34x-63 &\equiv \tfrac{1}{10}(400x^2+340x-630) \\ &\equiv \tfrac{1}{10}[(20x+\tfrac{17}{2})^2-\tfrac{28\,0\,2}{4}] \\ &\equiv \tfrac{1}{10}[(20x+\tfrac{17}{2})^2-(\tfrac{53}{2})^2] \\ &\equiv \tfrac{1}{10}(20x+35)(20x-18) \\ &\equiv (4x+7)(10x-9). \end{aligned}$$

This method also is rather cumbersome, but provides useful practice in another very important type, and in completing a square (soon to be needed in solving a quadratic equation). It provides amusement, I find, in the revision year; that is, when they have thoroughly mastered the easier method.

May I add that, in revising, we classify the types according to the number of terms? Apart from numerical or literal factors (too often overlooked) we look for the following:

I. 2 terms: Cf. x^2-y^2 or $x^3\pm y^3$.

II. 3 terms: (i) Cf. x^2+px+q.

(ii) Cf. ax^2+bx+c; express as 4 terms.

III. 4 terms: (i) Group "2 and 2"; find binomial factor.

(ii) Group "3 and 1" or "1 and 3", and reduce to type I.

While this does not indicate the order of teaching, or of discovery, it does present most of the types they are likely to meet in a systematised form which appeals to most boys (I cannot speak for girls).

D. T. R.

1608. *On Note* 1536.

There are two points in Mr. T. H. Foster's note to which I should like to refer: these are his paragraph II (i), and his method of dealing with the trinomial with non-unit coefficient of x^2.

Suppose it is desired to find factors of 2520 which differ by 34. A pair of obvious factors is 40 and 63; the difference here is 23,

which is too small, so that another pair must be found such that the greater factor is greater than 63 and consequently the smaller less than 40. Factorising 40×63 into $5 \times 8 \times 7 \times 9$ we get a hint as to possible factors less than 40; we try at random 35 and 72, but here the difference is 37, just too big. So we try 36 and get the other factor 70, which is just what we want, and we have made each choice of factors guide us to the correct answer.

Now for factorisation of $40x^2 + 34x - 63$ I use the following method. I ask myself what factors of the product 40×63 *differ* by 34 (because the term 63 is *minus*). By the method just described I get 36 and 70. I choose either and write down the following expression

$$(40x \quad 70),$$

where the index 2 has been dropped from the quadratic term. I next remove all common factors of these two numbers and get

$$(4x \quad 7).$$

I add to this the factor needed to make the two extreme terms right :

$$(4x \quad 7)(10x \quad 9).$$

I then insert the signs to make the x-term right :

$$(4x + 7)(10x - 9).$$

I. FITZROY JONES.

1609. *The circle in homogeneous coordinates.*

In *Gazette* No. 262 (*December*, 1940), which did not reach the writer till April 24, 1941, Mr. N. M. Gibbins takes objection to the use made of imaginary and infinite elements in *Note* 1448. He gives an alternative method of arriving at the results of the Note. It does not, however, appear to be in keeping with the spirit of coordinate geometry to make frequent appeals, as Mr. Gibbins does, to theorems in Pure Geometry. The basic foundation of the circle is its definition and the natural method of deducing its equation in any system of coordinates must depend on the expression for the square of the distance between two points whose coordinates are given. In the *areal* system, if (x, y, z), (x_1, y_1, z_1) be the *absolute* coordinates of two points P, P_1 respectively, we have

$$PP_1^2 = -\Sigma a^2(y - y_1)(z - z_1),$$

where a, b, c are the lengths of the sides of the triangle (ABC) of reference. (*Cf.* Askwith's *Coordinate Geometry*.)

2. *To show that the areal equation of a circle can be reduced to the form*

$$S \equiv (lx + my + nz)(x + y + z) - \Sigma a^2 yz = 0,$$

where l, m, n are the powers of A, B, C respectively, with respect to the circle.

If P, P_1 are two points as in § 1, then the power ϖ_P of P with respect to the circle whose centre is P_1, and radius ρ, is given by

$$PP_1^2 - \rho^2 = -\Sigma a^2(y-y_1)(z-z_1) - \rho^2(x+y+z),$$
$$= -\Sigma a^2yz + \Sigma(b^2z_1 + c^2y_1 + k)x,$$

where $k = -\Sigma a^2y_1z_1 - \rho^2.$

Hence $\varpi_A = AP_1^2 - \rho^2 = b^2z_1 + c^2y_1 + k,$

$\varpi_B = BP_1^2 - \rho^2 = c^2x_1 + a^2z_1 + k,$

$\varpi_C = CP_1^2 - \rho^2 = a^2y_1 + b^2x_1 + k.$

Hence $\varpi_P = -\Sigma a^2yz + \varpi_A x + \varpi_B y + \varpi_C z,$

which can be expressed as a homogeneous function of x, y, z in the form

$$S' = (\Sigma\varpi_A x)\Sigma x - \Sigma a^2yz.$$

Hence the equation of the circle (P_1, ρ) is $S' = 0$, which is of the form $S = 0$, where $l = \varpi_A$, $m = \varpi_B$, $n = \varpi_C$.

Corollaries :

(i) If l, m, n are the powers of A, B, C respectively with respect to a circle, then the power of $P(x, y, z)$, where $\Sigma x = 1$, is S, and the equation of the circle is $S = 0$.

(ii) The equation of the circle ABC is $\Sigma a^2yz = 0$.

(iii) The locus of points the powers of which with respect to the circles $S = 0$ and $\Sigma a^2yz = 0$ are equal, is the straight line

$$lx + my + nz = 0,$$

which is therefore the radical axis of the two circles.

3. *To find the centre and radius of the circle with respect to which the powers of A, B, C are l, m, n respectively.*

If $P_1(x_1, y_1, z_1)$, where $\Sigma x_1 = 1$, is the centre, and ρ the radius of the circle, then as in § 2, we have

$$b^2z_1 + c^2y_1 + k - l = 0, \quad \ldots\ldots\ldots\ldots(1)$$
$$c^2x_1 + a^2z_1 + k - m = 0, \quad \ldots\ldots\ldots\ldots(2)$$
$$a^2y_1 + b^2x_1 + k - n = 0, \quad \ldots\ldots\ldots\ldots(3)$$
$$x_1 + y_1 + z_1 - 1 = 0, \quad \ldots\ldots\ldots\ldots(4)$$

where $k = -\Sigma a^2y_1z_1 - \rho^2. \quad \ldots\ldots\ldots\ldots(5)$

Multiplying (1) by x_1, (2) by y_1, (3) by z_1 and adding the results, we have

$$2\Sigma a^2y_1z_1 + k\Sigma x_1 - \Sigma lx_1 = 0.$$

Hence, equation (5) can be replaced by

$$lx_1 + my_1 + nz_1 + k = -2\rho^2. \quad \ldots\ldots\ldots\ldots(6)$$

(i) *To find the centre* :

Employing the method of determinants to solve the equations (1), (2), (3), (4) for x_1, y_1, z_1, k, we get

$$\begin{vmatrix} 0 & c^2 & b^2 & 1 \\ c^2 & 0 & a^2 & 1 \\ b^2 & a^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} x_1 = - \begin{vmatrix} c^2 & b^2 & 1 & l \\ 0 & a^2 & 1 & m \\ a^2 & 0 & 1 & n \\ 1 & 1 & 0 & 1 \end{vmatrix}.$$

Hence

$$8\Delta^2 x_1 = a^2bc\cos A - a(al - bm\cos C - cn\cos B). \quad \ldots\ldots\ldots(7)$$

Similarly for y_1 and z_1.

Corollary :

If x_0, y_0, z_0 be the *absolute* coordinates of the centre of the circle ABC,

$$8\Delta^2(x_1 - x_0) = -a(al - bm\cos C - cn\cos B). \quad \ldots\ldots\ldots(8)$$

Similarly for $y_1 - y_0$ and $z_1 - z_0$.

(ii) *To find* ρ^2 :

Eliminating x_1, y_1, z_1, k from the equations (1), (2), (3), (4), (6), we have

$$\begin{vmatrix} 0 & c^2 & b^2 & 1 & l \\ c^2 & 0 & a^2 & 1 & m \\ b^2 & a^2 & 0 & 1 & n \\ 1 & 1 & 1 & 0 & 1 \\ l & m & n & 1 & -2\rho^2 \end{vmatrix} = 0.$$

Hence

$$16\Delta^2\rho^2 = -\tfrac{1}{2} \begin{vmatrix} 0 & c^2 & b^2 & 1 & l \\ c^2 & 0 & a^2 & 1 & m \\ b^2 & a^2 & 0 & 1 & n \\ 1 & 1 & 1 & 0 & 1 \\ l & m & n & 1 & 0 \end{vmatrix}$$

$$= \Omega - \Sigma a^2(b^2 + c^2 - a^2)l + a^2b^2c^2$$

$$= \Omega - 2abc\Sigma al\cos A + 16\Delta^2R^2, \quad \ldots\ldots\ldots(9)$$

where $\Omega \equiv \Sigma a^2l^2 - 2\Sigma bcmn\cos A$.

Corollary :

$$\partial\rho^2/\partial l = -x_1, \quad \partial\rho^2/\partial m = -y_1, \quad \partial\rho^2/\partial n = -z_1. \quad \ldots\ldots\ldots(10)$$

4. The results of § 3 can be obtained easily without the use of determinants. For, from equations (1), (2), (3), (4) of § 3, it is obvious that x_1, y_1, z_1, k are linear functions of l, m, n ; so that x_1 is of the form $\lambda l + \mu m + \nu n + p$; where p, λ, μ, ν can be obtained in succession by considering the particular cases of the circle ABC, and the circles on BC, CA, AB as diameters. This process determines $x_1 - x_0$ (*cf.* equation (8)). Similarly for $y_1 - y_0$ and $z_1 - z_0$.

To find ρ^2 : Equations (1), (2), (3), (4) of § 3 give

$$\Sigma lx_0 = \Sigma x_0(b^2z_1 + c^2y_1 + k)$$
$$= \Sigma x_1(b^2z_0 + c^2y_0) + k.$$

But $b^2z_0 + c^2y_0 = c^2x_0 + a^2z_0 = a^2y_0 + b^2x_0 = 2R^2.$

Hence $\Sigma lx_0 = 2R^2 + k.$ .....(11)

Now eliminating k from equations (6) and (11), we get

$$2R^2 - 2\rho^2 = \Sigma l(x_1 + x_0)$$
$$= \Sigma l(x_1 - x_0) + 2\Sigma lx_0.$$

Hence, with the aid of equation (8), we get

$$\rho^2 = R^2 - \Sigma lx_0 + \Omega/16\Delta^2.$$

F. H. V. GULASEKHARAM.

1610. *On Note* 1542 (*Gazette*, XXV, October 1941, p. 242).

If $y^3 + 3xy + 2x^3 = 0$, then

$$x^2(1 + x^3)\frac{d^2y}{dx^2} - \frac{3}{2}x\frac{dy}{dx} + y = 0.$$

Would Professor Neville prefer the following method?

Put $v = y/x,\ t = 1/x.$

Then the algebraic equation becomes

$$v^3 + 3tv + 2 = 0,$$ .....(i)

and the differential equation becomes

$$2(t^3 + 1)\frac{d^2v}{dt^2} + 3t^2\frac{dv}{dt} - vt = 0,$$ .....(ii)

which may be written

$$\frac{d}{dv}\left\{(t^3 + 1)\left(\frac{dv}{dt}\right)^2\right\} - vt = 0.$$ .....(iii)

It may now easily be verified that (i) satisfies (iii), since by (i)

$$t^3 + 1 = -(1 - v^3)^2(8 + v^3)/27v^3,$$

and $dt/dv = 2(1 - v^3)/3v^2,$

whence (iii) becomes

$$\frac{d}{dv}\{\tfrac{1}{4}(8v + v^4)\} + 3vt = 0,$$

which is the same as (i). F. BOWMAN.

1405. SIMPLE ENOUGH.—*Learned Professor :* "I can not understand how people forget children's ages—I have no trouble. For instance, I was born 2,320 years after Socrates ; my wife 1,900 years after the death of the great Caesar ; my son 1,500 years after the beginning of the Folk Wandering—quite simple you see!"—*The Passing Show.* Quoted in *Literary Digest.* (Per Mr. T. R. Dawson.)

REVIEWS.

The analytical foundations of celestial mechanics. By AUREL WINTNER. Pp. xii, 448. 36s. 1941. Princeton Mathematical Series, 5. (Princeton University Press; Humphrey Milford)

Between the plain statement of a problem in dynamics and the complexity of its solution there is commonly a remarkable contrast. Even the simple pendulum is not quite as simple as it may have appeared to Galileo. A brick set turning and left to itself in empty space adds only geometrical complications, but after a few well known and highly special cases the more general problem of the top soon reaches an intractable shape. And nothing can be easier than to imagine three massive particles projected in space and left to their mutual attraction. Yet here is a problem the complexity of which bids fair to be inexhaustible.

The situation is noteworthy because as a general rule, after a problem has been studied for a couple of centuries and more by the greatest minds, some approach to finality is to be expected. In this instance the further the advance, and that is won painfully enough, the further the end recedes into an unending vista. But there is a distinction to be made according to the point of view, that of the astronomer and that of the mathematician. The separation in itself is symptomatic of modern tendencies. For the astronomer in this sense does not mean necessarily an observer, but one whose primary concern is with the relation between observation and theory, and that involves a technique and a tradition, built up painfully from the days of Clairaut and Euler. In this sense Laplace was an astronomer while Lagrange as certainly was not, if there be any truth in this reported complaint, made on leaving a meeting of the Paris Academy of Sciences: these astronomers are strange people, they won't let any conclusion pass unless it agrees with their observations.

The author of the present work has the advantage of combining both points of view. He is well aware that "ingenious and perfectly satisfactory considerations often occur in publications which were not written by ϵ-trained mathematicians". At the same time he has a wide command of the methods of modern pure analysis and is concerned in this book with their introduction into theoretical dynamics and the theory of perturbations. But the opposite fear obtrudes, and (to quote again from his preface): "I have tried not to repel that regrettable majority of younger mathematicians who have had no contact with theoretical astronomy". Between the two deflecting influences he has succeeded in writing a very interesting book.

The contrast between the astronomer and the pure mathematician has grown wide and gives no immediate promise of closing. On his side the astronomer, with his practical object in view, has made great strides, and has not only provided largely for his own needs but has also furnished the classical exercise ground in dynamics and even in pure analysis. The astronomer from the time of Clairaut has been compelled to go forward aiming at a solution by quadratures if possible, but more generally being satisfied with methods of successive approximations or even the numerical methods of finite differences. He must be content with restricted intervals of time, he is glad when justification comes from the mathematician and naturally welcomes all the aid to be derived from the same quarter, for his task is a hard one. In the course of it he has raised and of necessity passed over to other hands unlimited questions which go far deeper into the real problem, *e.g.* the question of stability. It is these questions with which the author deals in the main, and perhaps a fair idea of the nature of his book is suggested by noting the following passage (p. 345): "As seen from § 200-

§ 201, every new generation usually is compelled to reinterpret what *the* problem of three bodies actually is. Until Birkhoff realised and further developed Poincaré's geometrical ideas concerning dynamical systems with two degrees of freedom, the answer to the question used to be this : On the one hand, the problem of three bodies cannot be ' solved ', in view of the established non-existence of integrals of specific type (§ 129, §320 *bis*) ; while, on the other hand, the problem of three bodies may be considered as ' solved ', in view of the convergence of certain expansions, established along the whole t-axis (§ 432 *bis*). To-day one is inclined to consider the first of these statements as inadequate, and the second as quite meaningless, and accordingly to formulate the problem of three bodies in terms of an ' incompressible flow ' on a seven-dimensional manifold, . . ."

This will serve to indicate that the work is addressed to the mathematician with a modern equipment rather than to the astronomer with theoretical leanings, though it may help to bridge the gap between the two. To this end the Historical Notes and References added as an appendix should be as valuable as they are interesting, and this feature causes the more regret that the book lacks a name-index, since the subject-index fails to supply the want.

H. C. P.

Advanced Calculus. By I. S. Sokolnikoff. Pp. vi, 466. 26s. 1939. (McGraw-Hill)

Advanced Calculus. By C. A. Stewart. Pp. xviii, 523. 25s. 1940. (Methuen)

These books give different impressions ; each has its own advantages. The former can be read in two ways. It can be read fairly rapidly ; or the reader can force himself to study it carefully from which application he would receive much benefit. In some ways it recalls the style and scope of de la Vallée Poussin's *Cours d'Analyse*, if the geometry were omitted from the latter. There are few points which call for comment. There are a few points which the reviewer thinks could have been clearer. The chief of these are that the theorem on convergence of monotonic sequences should have come earlier, and that bounded variation should have been assumed instead of Dirichlet's conditions for the convergence of Fourier series ; also the essential point of Dirichlet's conditions only appears 19 pages after that discussion.

Only one real error was noticed. In a suggestion for the reader to find a series for the calculation of the complete elliptic integral of the first kind when k is near unity, it will be found that the argument suggested assumes the convergence of $\int_0^{\pi/2} \tan^{2n}\theta \sec\theta\, d\theta$ for all positive integers.

The book will be of value to the student who seriously applies himself to everything said and suggested in it (with that one exception).

The first opening of the other book gives a different impression. It is at once obvious that close reading is called for ; and, very soon, that the student who is willing to give this will be well repaid. The range of the book is wide ; it covers a large part of both real and complex variable theory and includes many applications. The reader will not only gain a knowledge of general theory but will acquire power and confidence to deal with the calculations in particular problems. This makes it unnecessary to point out any of the remarkably small number of misprints that have been noticed, or one or two points which would be obscure on the first reading ; the student can come back to them after a while and find that he has learnt sufficient from the book to clear them up himself.

Some guidance will be needed by a pupil who may be put off by the range of the book being greater than his immediate needs, and every pupil will need

guidance in making a suitable selection from the very large number of examples. Neither of these should be allowed to discourage the use of the book, which can be strongly recommended. R. C.

Transients in electric circuits, using the Heaviside operational calculus. By W. B. COULTHARD. Pp. vii, 203. 25s. 1941. (Pitman)

Heaviside introduced his operational method about the year 1893, Bateman used what may be called the pe^{-pt} method in 1910, but did not pursue the matter nor couple it with operational methods, and Bromwich introduced the contour integral approach in 1916. For many years engineers followed Heaviside, although they worked in the dark and hoped for the best. Mathematicians followed Bromwich, but at times with doubts in their minds. In the past decade the situation has been clarified, and it is now recognised that fusion of the pe^{-pt} and contour integral methods, as in the application of the Mellin inversion theorem, enables certain types of linear differential equations to be treated unambiguously. Those who have been reared on the Heaviside plan do not wish to see their operational conjuring tricks displaced by mere mathematics. Consequently there is a school of operational diehards, which will not be extinct for some years to come. Until the Mellin theorem approach is accepted universally, it will be necessary to introduce complex integration in small doses to engineering and technical faculties in universities and colleges throughout the English-speaking world. The subject will be branded as difficult by the diehards, and also by those with little liking for mathematics. About half a century ago the calculus was considered by many engineers to be beyond their reach, whilst some mathematicians regarded the application of one of their important tools to practical affairs as a sacrilege. It is mainly a question of mental attitude, and as this must change ultimately, why not change now and save valuable time?

With the above facts in mind, we consider the present book, which has been written for electrical engineers. The author commences in the appropriate Heaviside way, and the reader is left guessing as to what p really means. The equation $Ri + L\,di/dt = E$ is solved in terms of t and p, and by comparison of terms, the operational equivalences $p^{-1} \doteqdot t$, $p^{-2} \doteqdot t^2/2$, etc., are obtained. This is followed by the expansion theorem. Although complex integration is not used, references are made to Bromwich and Jeffreys at the end of the first chapter. The Laplace integral is credited to Carson, but this is a common error and occurs in many books and papers. Carson clearly states in his book that the integral

$$\int_0^\infty e^{-pt} f(t)\,dt$$

is due to Laplace. On the whole the treatment in the book under review cannot be said to be satisfactory from a mathematical standpoint. To anyone accustomed to the clear logic of mathematics, it is difficult to follow. Some of the statements concerning the " forms " of certain Bessel functions in Chap. 11 are incorrect. For example $I_0(0) \neq 0$, and $I_0(x)$ does not behave as e^x but as $e^x/\sqrt{x}$ when x is large.

However, the principal feature of the book is not its mathematics, but the solution of problems in various branches of electrical engineering. The choice of material is good and covers a wide and interesting field. It is tacitly assumed that the reader has a practical working knowledge of the electrical apparatus discussed. Some of the subjects treated are: oscillographs, transformers, circuit-breakers, direct and alternating current dynamos and motors, transmission lines, filter circuits, and eddy currents in armature and transformer plates. These are all of considerable importance to electrical engineers.

L

The last chapter deals with electrical circuits having variable resistance, capacitance and inductance. The equations are solved by successive approximation and by graphical methods, as there is no general method of approach. A number of useful references is given at the end of each of the twelve chapters. The book closes with two appendices: one is a list of about 80 operational forms of various functions which occur in applications, the other is a brief list of sundry mathematical formulae. N. W. McL.

Elementary Aircraft Mathematics. By J. B. CHANNON and A. MCLEISH SMITH. Pp. 97. 1s. 6d. 1942. (Longmans)

This is a book of exercises for Air Training Cadets. They are well graded with plenty of easy ones and many of aviation interest. Each set is as a rule preceded by one or two worked examples. The explanatory matter is clear and concise. Except for Areas and Volumes the book covers the A.T.C. syllabus exactly. Answers are provided. R. C. L.

Mathematics for Technical Students. III. By A. GEARY, H. V. LOWRY and H. A. HAYDEN. Pp. viii, 371. With tables. 8s. 1941. (Longmans)

This is the concluding volume of a book intended to cover a three years' national certificate course in mechanical or electrical engineering, and it maintains the high standard set by the first two volumes, which were reviewed in the *Math. Gazette*, Vol. XXIII, p. 112, and Vol. XXIV, p. 138. Many books are published with titles similar to that of this one, but few succeed so admirably in meeting the needs of technical students, while at the same time developing the necessary mathematics with as much rigour as that class of student can appreciate. Where a result is needed which it is beyond the scope of the book to prove adequately, the authors do not hesitate to say so, and this attitude is much to be preferred to the giving of unsound proofs.

The book starts with a revision chapter consisting almost wholly of a large number of excellent examples, all of which have a distinctly utilitarian flavour. Chapters on series, graphs and trigonometry follow, with many excellent applications in which the engineering student will find that his particular point of view is well catered for. Most of the rest of the book is devoted to calculus, the differentiation of all the elementary functions, and the corresponding simple integrals being dealt with; all the usual applications are again well illustrated with examples, and many quite difficult technical problems are approached from a simple point of view. The chapter on complex numbers will be of interest mainly to the electrical engineer, who will find that his own notation is used throughout. The book concludes with a chapter on alignment charts and three specimen examination papers of the standard for which it caters. These show that the book covers much more than appears to be normally required, but this is rather an advantage than otherwise.

Only a few adverse criticisms can be made. Some series are summed by direct addition and it is tacitly assumed that when a term is arrived at which does not affect the last decimal to be retained, the rest of the series may be safely neglected. Some warning that this is not always true should be given; engineering students are much too apt to cut series short. It would not be beyond the capacity of the student to appreciate the idea of an upper bound to the remainder in all the cases which are dealt with. The formula for the derivative of x^n is at first proved for positive integral values of n only though the result is freely used for fractional and negative values, and the general result is not referred to again until the exponential function has been dealt with. Though the authors are usually very careful about the idea of a limit, an infinite integral appears on p. 167 as an exercise, without any previous

definition, and on p. 262 the student is asked to find the area bounded by the curve $y=e^{-x}$, the y-axis and the positive half of the x-axis, no attempt being made to define the meaning of "area" in such a case. Some mistakes in the answers to exercises have been noticed; those to example 2, p. 188, and example 27, p. 199, are not wholly correct.

These are, however, no more than minor blemishes in a book which deserves a wide circulation. N. M. H. LIGHTFOOT.

Early military books in the University of Michigan Libraries. By T. M. SPAULDING and L. C. KARPINSKI. Pp. xvi, 45; 37 plates. $2. 1941. (University of Michigan Press, Ann Arbor, Michigan)

The Rouse Ball Professor of Mathematics (Oxford), when President of the London Mathematical Society, presented the De Morgan Medal of the Society to the Rouse Ball Professor of Mathematics (Cambridge) and reminded those present that at one time, namely 1914-1918, Professor Littlewood's researches were devoted to what might be called applied mathematics. Few mathematicians alive to-day can be unaware of the importance of their subject in modern warfare. This may explain the appearance in these columns of a review of the present book.

The University of Michigan is fortunate in possessing a large collection of military works, many of which have been given by Colonel Spaulding, U.S. Army, joint-author of the list under review. This list deals with books published up to 1800 and contains 372 titles. That enthusiastic and able bibliographer, Professor L. C. Karpinski, has collaborated in producing a volume which should enable American scholars to identify and locate in one or other of the Libraries of the Union many interesting and valuable works. The plates reproduce about 120 title-pages; the reproductions are excellent and add greatly to the interest of the list.* Thus, one plate shows us Galileo's *Discorsi*, the Lord of Praissac's *Short method for the easie resolving of any militarie question propounded* (perhaps the Home Guard should note), Robert Ward's *Animadversions of Warre*, and *The Complete Captain, or, An Abbridgement of Cesars warres*, . . . by the Duc de Rohan.

In the list we notice many familiar names: Thomas Digges, Ozanam, Mersenne, Tartaglia, Stevin, and (somewhat to this reviewer's surprise) Legendre. T. A. A. B.

Table of squares of all numbers less than 300. 1s. 1942. (Scientific Computing Service, 23 Bedford Square, London, W.C. 1)

So many computations demand the use of squares of integers that this card, issued by Dr. L. J. Comrie's enterprising Computing Service, should be extremely convenient and popular. T. A. A. B.

Mathematical Monographs. I. Edited by D. R. CURTISS. Pp. vii, 172. $2.25. 1941. Northwestern University Studies in Mathematics and the Physical Sciences, 1. (Northwestern University, Evanston and Chicago)

This small volume contains four monographs:

(i) Maxima and minima of functions of two or more variables, by D. R. Curtiss;

(ii) The statistics of time series, by H. T. Davis;

(iii) Topics in continued fractions and summability, by H. L. Garabedian and H. S. Wall;

(iv) Spectra of quadratic forms in infinitely many variables, by E. D. Hellinger.

* On Plate XXVII, *for* 313 *read* 213.

These monographs are based on sets of lectures given at Northwestern University, and contain original work done by the respective authors. But in each case, an effort has been made to render the account self-contained and to present a reasonably complete picture of the small but well-defined domain under consideration.

Thus, for example, in (iii) we have a triangle of ideas; one side was laid down by Stieltjes in his paper "Recherches sur les fractions continues" in the *Annales de Toulouse* (1894) in which he connected continued fractions with his now famous moment problem; the second side was added by Hausdorff in two powerful papers, "Summationsmethoden und Momentfolgen", in the *Mathematische Zeitschrift* (1921), in which the matrix theory of summation of divergent series was applied to the moment problem. The present authors discuss briefly but clearly these two sides and complete the triangle by linking continued fractions with summation processes.

The student beginning research work on any topic related to one of these four should find the book most helpful. T. A. A. B.

Differential equations. By G. S. Diwan and D. S. Agashe. Pp. x, 316. N.p. 1941. (St. Xavier's College, Bombay)

Forsyth, Piaggio and Ince have set such a high standard of exposition in books in English on differential equations that comparison with these of textbooks which are perhaps less ambitious is inevitable. The present work is nearest in scope to Piaggio's, but it contains nothing about the existence of solutions and very little on partial differential equations, the presumable and adequate reason being that it is intended for use in Indian Universities and therefore restricts itself to the appropriate syllabus. The exposition does not rival Piaggio's convincing clarity, and tends to be long-winded. This is to some extent intentional, as the authors set out to deal with the subject, they say, in a manner less incomplete than that of some of the textbooks in common use in India. In places, their painstaking care has been successful; in other places we feel that we might understand better if the authors had not explained so much. Thus, for example, a great deal of space is taken up with the investigation of particular integrals of linear equations with constant coefficients, using the operator $1/f(D)$, but definitions and questions of uniqueness are not very satisfactorily expounded and comments intended to be helpful are not always clear, for example (p. 99): "generally an operational expression, wherein the operator comes into contact with itself and constants, obeys all laws of Algebra even when it continues to have its operative significance".

Many examples are worked out fully, and many more are given for the student, most of which come from examination papers set in Indian Universities. T. A. A. B.

The Four Color Problem. By Philip Franklin. Pp. 33. 50 cents. 1941. (*Scripta Mathematica*, Yeshiva College, New York)

This booklet explains some of the intricacies of the above problem in a succinct and interesting manner. The main question at issue is whether the divisions of *any* ordinary map can be properly distinguished by the use of four colours only, so that no two adjacent divisions bear the same colour. And it is still a problem, because the challenge thrown out to mathematicians by Professor Cayley, more than sixty years ago, to give a valid proof that this is so, has not yet been satisfactorily met. An attempt put forward in early days by Mr. A. B. Kempe was found to have a fatal flaw.

An extensive literature has since gathered about the subject, which is very

adequately treated by Mr. Franklin within the compass of thirty-three pages. He gives the most important results, with proofs where they are simple, and references to original papers where they are long and tedious; so that the brochure is very readable. Compression has led to an obscure sentence here and there, and there is confusion in the references to two early papers in the *Quarterly Journal*; but on the whole the explanations are clear.

The proved results are of two kinds: some are such as are or might be subsidiary to the main proposition; *e.g.* Petersen's theorem that in a regular map, where all the corner points are meeting points of three divisions, a set of "edges", *i.e.* segments of boundary, can be found such that just one of their end points occurs at each corner point. (For this there is a condition—not very clearly stated.) On the other hand there are extensions and modifications of the main proposition, which—strangely—are much easier to prove. For instance, if all the divisions of a map are even, *i.e.* touched by an even number of others, the map can be coloured with 3 colours; if all are triple, *i.e.* numbers of sides all multiples of 3, then 4 colours can be proved sufficient. Again, on a multiply-connected surface the problem can be solved: on an anchor ring 7 colours may be necessary and they are sufficient for any map. Further, if on a simple surface each "region" may consist of two distinct portions which must be coloured alike, 12 colours are sufficient and may be necessary; so with r portions $6r$ colours will suffice.

Perhaps the most interesting side-light on the main problem is that the colouring of an ordinary map in 4 colours can be shown to depend on the possibility of distinguishing the "edges" in three ways so that all three kinds of boundary meet at each corner point; and this again depends on the possibility of placing + or − at each corner point, so that the difference between the number of +'s and −'s round each division is a multiple of 3. The problem is thus reduced to that of the solution of a system of linear congruences, one for each division:

$$x_1 + x_2 + \ldots + x_r \equiv 0, \mod 3;$$

where the x's (one for each corner point) are restricted to the values +1 and −1 (or 2), not zero. This is noticed by Mr. Franklin, who says it is not practical, even for simple maps; *i.e.* it is not the simplest way of arriving at a valid scheme of colouring. Its interest lies entirely in its reduction of a geometrical to an algebraical problem. He omits, perhaps wisely, to notice the extension of view to congruences of this type with 0, 1 or 2 on the right-hand side, not necessarily zero. Such systems come naturally for consideration, in that *any* assignment of +1 or −1 to the x's necessarily gives a solution of one such system. The question of what constants are possible, and of the reason for the few "failures" which occur in simple cases, has been extensively discussed. Sometimes a wide generalisation may throw light on what has then to be regarded as a very particular case.

Perhaps the most notable achievement, due to a succession of investigators ending with C. E. Winn, is the proof that the four-colour theorem cannot fail for an ordinary map *consisting of fewer than* 36 *divisions*. P. J. H.

Notes on the Regular Icosahedron and the Regular Dodecahedron. By LESTER S. HILL. Pp. 11. 10 cents. Reprinted from *Scripta Mathematica*. (Yeshiva College, New York)

If the legend of George Washington's youth is true, the motive for his early arboricide was not interest in the victim but in the instrument, his little hatchet; and in the same spirit his fellow-countryman, in applying the method of "semi-homogeneous co-ordinates" to the regular polyhedra, has

apparently had in mind a test of the power of his method rather than any addition to the world's knowledge of twelve- or twenty-sided solids.

t being the latitude and g the longitude of a point on a sphere, its "semi-homogeneous co-ordinates" are $x = k\cos g$, $y = k\sin g$, and $z = k\tan t$, k being any positive real number. (These are in fact the Cartesian coordinates of an arbitrary point on the radius through the point to be described by them.)

If the object of this paper, which seems to be one of a series, is to familiarise the reader with the method, that later he may understand its use in work for which it is better suited, then it may well serve its purpose in paving the way for what is to follow; but taken by itself, and considered as an essay on polyhedra, it is disappointing. The existence of these figures can be proved, and their measurements calculated, by far less ponderous methods: for instance, the perpendicular bisector of the meridian arc joining two points of the same longitude hardly needs such a steam-roller as

$$\frac{xx_1+yy_1+zz_1}{(x_1^2+y_1^2+z_1^2)^{\frac{1}{2}}} = \frac{xx_2+yy_2+zz_2}{(x_2^2+y_2^2+z_2^2)^{\frac{1}{2}}},$$

the general formula for the perpendicular bisector of the arcs joining *any* two points on a sphere.

In the section on the 59 rotations which will bring a regular icosahedron into the same portion of space from which it started, the mass of notation introduced is disproportionate to the simple results obtained; but here again it may be that the hatchet is being tried out on twigs before the attack on the forest begins.

W. H.-J.

The Structure of the Honeycomb. By HARRY POLACHEK. Pp. 12. 20 cents. Reprinted from *Scripta Mathematica*. (Yeshiva College, New York)

In the first third of this slender but entertaining tract the author discusses the geometry of the bee's cell, with special reference to the Minimum-area property, that is, the inclination of the rhombi separating East-facing from West-facing cells at the angle which gives the requisite volume with the least expenditure of wax.

In the rest of it he gives us a résumé of the storm of controversy which has raged about the question, What enables the bee to build a structure so admirably adapted to its purpose? Moderation, a quality which we expect of mathematicians, has not characterised many of the disputants: Bigelow, for instance, proclaimed that "the bee is as little responsible for the hexagonal form as she is for the movements of a planet", to which Heuffaker answered, "Mr. Bigelow has shown himself equally ignorant of bees, mechanics and history". *

Moraldi, of Paris Observatory, claimed to have *measured* the angles under discussion *to the nearest minute*; but Paris, we may remember, has always been famed for its cookery.

Darwin, as usual, gives a reasonable explanation, neither postulating Archimedean intelligence in the bee nor minimising what he calls "the most wonderful of all known instincts", while François Huber finds evidence for attributing to the bee some power of thought rising above the instinct-level.

But to British mathematicians the item of chief interest in this history will be that in 1743 Maclaurin proved *by pure geometry alone* that the minimum area is given by taking the acute angles of the rhombi and trapezia as $\sec^{-1} 3$, which is actually the angle to which the bee makes a very creditable approximation.

W. H.-J.

* Perhaps our modern suffix-maniacs would say that ${}^{B}I_b = {}^{B}I_m = {}^{B}I_h$.

Musical logarithms. By J. M. BARBOUR. Pp. 11. 10 cents. Reprinted from *Scripta Mathematica*, VII, pp. 21-31. 1940. (*Scripta Mathematica*, Yeshiva College, New York)

e and 10 are not the only logarithmic bases for which practical use has been found. The musician's perceptual measure of pitch is related logarithmically to the "frequencies" of the physicist and thus it happens that logarithms of frequencies provide the most convenient means of specifying the notes of a musical scale. The Pythagorean "comma" (approximately 1/53 of an octave) was the earliest common measure proposed, but if decimals are to be used, the octave itself is a more natural base. A table is reprinted from Caramuel (1670) giving musical logarithms for 3 octaves on this system (*i.e.* base 2). Equal temperament suggests 1/12 of an octave, and Lambert used this base in 1774, working to 15 decimal places. If a scale is measured on this system the divergences of its notes from those of the "well-tempered clavichord" are immediately apparent. In modern acoustics the "cent" (1/1200 of an octave) is used. E. H. L.

Intermediate Algebra. By NEIL MCARTHUR and ALEXANDER KEITH. Pp. x, 356. 8s. 6d. 1942. (Methuen)

This book bears a title which is aptly descriptive of the contents. The standard of the work is intermediate between that of the first school certificate and that suitable for those likely to specialise in Mathematics; while the book covers just those topics in Algebra which can be reasonably expected at such a stage. Equations; indices, surds, and logarithms; Partial Fractions; Binomial and other series; permutations and combinations; the method of differences; and approximations are all adequately treated. Two chapters on Graphs, with exceptionally clear diagrams, deal fully with this portion of the work. The whole is well arranged and presented; interest is added by a number of historical references; and there is a considerable variety of worked examples. The sets of questions are provided with accurate answers; and there are three pages of "Index".

The only general criticism that may be offered is that some of the treatment is rather more than adequate: that in order to achieve a "self-contained" arrangement some matter appears which might be considered superfluous (*e.g.* the formula used in the solution of quadratic equations is derived, and not merely stated). Such portions are, of course, valuable for revision.

Similarly, one wonders whether the full sections on gradient of a line and of a curve, and the associated portions such as that on Limits, are not likely to be made unnecessary by the student doing similar work, probably to a higher standard and with greater rigidity, in other portions of his course. This is no plea for sectionalisation; it is naturally better to study (and to teach) Mathematics than "bits and pieces"; and the days of watertight compartments are over. Indeed it may be for these very reasons that the authors have included the sections referred to; for there are also notes on Complex numbers and a Calculus Proof of the Binomial Theorem.

The text, generally, reveals a more precise and adult approach than is usual in non-advanced books. Two quotations may indicate the style. On page 18 occurs the paragraph: "A function of a variable may have a geometrical or physical origin which permits its values to be obtained, or reasoned upon, when no analytic expression to furnish them is known. As the function may be regarded thus as having an independent existence we have to envisage the possibility of its values being obtainable from more than one such expression. Therefore the fundamental test that two mathematical expressions represent the same function must be identity of values rather than identity of form."

Again, on page 120 we have: "Logarithms to the base 10 are called *common* logarithms. For the most part they are irrational numbers, so that rational approximations must be used, . . . The results of the calculations are therefore only approximate. . . . This disadvantage would be present in any system. (On the other hand, *calculating machines* give *exact* results over a reasonably wide range, and for this and other reasons they are ousting logarithms for certain purposes.)"

One may expect this work to bring considerable credit to the authors, who are Lecturers in Mathematics at Glasgow, one in the University, the other in the Centre for the Training of Teachers.

The part played by the publishers should also be mentioned. The book is very clearly printed on good paper and the price must be deemed moderate —the more so in this time of ever-rising costs. F. W. K.

An Introduction to Analytical Geometry and Calculus. By T. K. RAGHAVACHARI. Pp. xx, 192. Rs. 2. 1941. (Oxford University Press, Madras)

This book disappoints in one respect only: the title might lead one to hope that a presentation concurrently of elementary calculus and coordinate geometry had been successfully made. Unfortunately this is not so. The volume comprises, in effect, simply the two sections bound together although the pagination, chapter numbers, etc., are continuous throughout. The demarcation is emphasised by the placing of the answers to the coordinate geometry questions on pages 92 to 96, before the start of the calculus text. Further, there is hardly any cross-reference from one portion to the other. Opportunities seem to have been missed here; *e.g.* the equation of the tangent to the circle could well have been obtained as an "application of differentiation", and so on.

But, this having been said, the rest must be entirely praise. The two portions, considered separately, are excellently devised and clearly written. There are, as a foreword by Professor Kibble of Madras Christian College says, "points of practical application which are bound to add to the students' interest in the subject, and evidence of original ideas and thorough thinking on the questions that arise in seeking to make quite clear the fundamental ideas".

The students for whom the book is intended are those taking Intermediate courses of South Indian universities, and a number of corresponding examination questions are included. The coordinate geometry involved is that of the straight line and circle. The calculus deals with limits (where the various theorems are mainly given without proof), differentiation, integration, and straightforward applications.

Two items may be noticed particularly: first, that the idea of a limit and the relations involved are taken as "almost intuitively evident"; and secondly, the interesting eight-page introduction which gives a very fair summary of the outstanding points in the history of the subjects, and some comments on the essential nature of Mathematics in general.

There is a full list of contents, but no index. The printing (in India) is good and accurate. I have worked (or caused to be worked!) almost all the many examples; and have found no error. F. W. K.

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